MATHS 203 LITE

solved past questions by Yuguda & Calculus

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SOLUTIONS TO 2018/2019 EXAMINATIONS QUESTION ON REAL ANALYSIS

Q1a(i) Define a convergent sequence and prove that the limit of a convergent sequence is unique

ANSWER

A sequence (x_n) of a real number is said to be converge (or is called a convergent sequence) if $\forall \varepsilon > 0 \; \exists$ a natural number $N(\varepsilon)$ depending on ε) $\ni$

$|x_n - x| < \varepsilon \quad \forall n \geq N(\varepsilon)$

x is called the limit of the sequence $\{x_n\}$

Q1aii prove that the limit of a convergent sequence is unique

ANSWER

The limit is unique i.e (if $x_n \to x$ and $x_n \to y$ as $n \to \infty$ then $x = y$)

Suppose by way of contradiction that the sequence $\{x_n\}$ converge to two limits x and $y \ni x \neq y$

Then $\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in N$

$\ni |x_n - x| < \frac{\varepsilon}{2} \quad \forall n \geq N(\varepsilon)$.

Then $\exists N(\varepsilon) \in N \ni |x_n - y| < \frac{\varepsilon}{2}$

$\forall n \geq N(\varepsilon)$. Hence

$\forall n > \max\{N(\varepsilon), N(\varepsilon)\}$, both inequalities holds.

$|x_n - x| < \frac{\varepsilon}{2}$ and $|x_n - y| < \frac{\varepsilon}{2}$ hold

then $|x - y| = |x - x_n + x_n - y| \leq |x_n - x| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\forall n > \max\{N(\varepsilon)\}, \bar{N}(\varepsilon)\}$

it means $|x - y| < \varepsilon$

$x \to y$ as $n \to \infty$, $\lim x = y$

Hence $x - y = 0 \quad \therefore x = y$

Q1b (i)

State without prove the density property of a real number system

ANSWER

If x and y are real number with $x < y$ then $\exists$ a rational number $r \ni x < r < y$

Q2a(i)

Define a Bonded Sequence and prove that every Convergent Sequence is bonded?

ANSWER

A sequence is said to be bonded if it is bonded both from above and below, thus, we say that a subject S is said to be bounded.

If $\exists$ a constant $x > 0 \ni |x| \leq m$ $\forall x \in S$. i.e $|x| \leq m \; \forall x \in S$

Iff $-m \leq x \leq m \quad \forall x \in S$

Q2a (ii)

Every Convergent Sequence is bounded?

ANSWER

Let the sequence $\{x_n\}$ converges to x. Then by definition $\forall \varepsilon > 0$ $\exists N(\varepsilon) \in \mathbb{N} \ni |x_n - x| < \varepsilon \ \forall n \geq N(\varepsilon)$

Let $\varepsilon = 1$ and hence

$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$ $\forall n \geq N(\varepsilon)$

i.e $|x_n| \leq 1 + |x| \ \forall n \geq N(\varepsilon)$ —— (1)

It remains to show that

$|x_n|$! $n = 1, 2, 3, --- N(\varepsilon - 1)$ is bounded

i.e $\{|x_n| \leq n : n = 1, 2, 3, --- N(\varepsilon)\}$

but $K = \max\{|x_n| : n = 1, 2, [illegible] N(\varepsilon)\}$

Now for these;

$|x_n| \leq K$ for $n = 1, 2, 3, --- N(\varepsilon)$ —— (2)

from (1) and (2) we have

$|x_n| \leq 1 + |x| + K \ \forall n \geq 1$

i.e $|x_n| \leq m$ where $m = 1 + |x| + K$.

Hence $\{x_n\}$ is bounded.

Q2b

Define the supremum of set and Prove that $\sup\{\frac{n}{2n+1} : n = 1, 2, 3 ---\} = \frac{1}{2}$

Answer

Let S be a subset of a real number which is bounded above. The lowest upper bond of S (LUB) on the supremum of S is denoted by supremum S is a real number β_0. Satisfying the following Conditions:

(i) $s \leq \beta_0 \ \forall s \in S$

(ii) If $s \leq \beta \ \forall s \in S$ then $\beta_0 \leq \beta$

These two condition give rise to:

(i) β_0 is an upper bond of S

(ii) $\forall \varepsilon > 0, \exists s \in S \ni \beta_0 - \varepsilon < s \leq \beta_0$

prove $\sup\{\frac{n}{2n+1} : n = 1, 2, 3 ---\} = \frac{1}{2}$

it sufficies to show that the 1st condition (i), $\frac{1}{2}$ is an upper bond of S.

i.e $\left(\frac{n}{2n+1} \leq \frac{1}{2}\right)$

(ii) $\forall \varepsilon > 0, \exists s \in S \ni \frac{1}{2} - \varepsilon < s \leq \frac{1}{2}$

clearly $\frac{n}{2n+1} \leq \frac{n}{2n} < \frac{1}{2}$ [illegible] ≥ 1

So, $\frac{1}{2}$ is an upper bond of S.

To verify (ii), Let $\varepsilon > 0$ be abituary, let us now find $s \in S \ni \frac{1}{2} - \varepsilon < s \leq \frac{1}{2}$

Since $s \in S$, it must have the for $\frac{n_0}{2n_0 + 1}$ for some positive Integer number

i.e $\frac{1}{2} - \varepsilon < \frac{n_0}{2n_0+1} \leq \frac{1}{2}$.

but $\frac{n_0}{2n_0+1} \leq \frac{1}{2}$ hold

$\forall$ integer $n_0 > 1$

but $\frac{1}{2} - \varepsilon < \frac{n_0}{2n_0+1}$ hold

$(2n_0+1)(1 - 2\varepsilon) < 2n_0$

$2n_0 - 4n_0\varepsilon + 1 - 2\varepsilon \leq 2n_0$

$1 - 2\varepsilon < 4n_0\varepsilon$

$n_0 > \frac{1 - 2\varepsilon}{4\varepsilon}$

Hence $\sup\{\frac{n}{2n+1} : n = 1, 2, 3 ---\} = \frac{1}{2}$

Using $n^2\sqrt{\frac{1}{\frac{n^2}{n^2}-1}}$ → Polynomial

$$\left(1-\frac{1}{n^2}\right)^n\left(\frac{n}{n-1}\right) \quad n\geq 2$$

$$>\left(1-\frac{1}{n}\right)\left(\frac{n}{n-1}\right) \text{ From Bernoulli's Inequality}$$

$$=\left(\frac{n-1}{n}\right)\left(\frac{n}{n-1}\right)=1$$

Thus $\frac{x_n}{x_{n-1}}\geq 1$ and $\{x_n\}$ is increasing, therefore it is a monotone increasing.

Q4b

State and prove the Bernoullis Inequality?

ANSWER

It states that $\forall n$ let $p>-1$ and $p\neq 0$ then for every integer $n\geq 2$ we have $(1+p)^n > 1+np$

proof from Induction

let $n = 2$

$(1+p)^2 = 1+2p+p^2 > 1+2p$

so, it is true for $n=2$. Suppose it is true for $n=k$

L.H.S $(1+p)^k > 1+kp$

when $n = k+1$

L.H.S $(1+p)^{k+1} = (1+p)^k(1+p) > (1+kp)(1+p)$ since it is true for $n=k$

$= 1+kp+p+kp^2$

$= 1+(k+1)p+kp^2 > 1+(k+1)p$. R.H.S

so, it is true for $n = k+1$

Hence by Induction the Inequality is

Q5a

Define a Cauchy Sequence and prove that every convergent sequence is a cauchy Sequence.

ANSWER

Cauchy Sequence: A sequence (x_n) is called a Cauchy sequence (or is said to be cauchy) if $\forall \varepsilon>0$ $\exists n_\varepsilon \in \mathbb{N} \ni$ $|x_n - x_m| < \varepsilon$ $\forall n, m\geq n_\varepsilon$.

proof Let the sequence (x_n) converges to x, we want to show that the sequence (x_n) is Cauchy. $\exists n_\varepsilon \in \mathbb{N}$, Then

Let $\varepsilon>0$ be given

From definition of convergent

$|x_n - x| < \frac{\varepsilon}{2}$ $\forall n \geq n(\varepsilon)$

$|x_m - x| < \frac{\varepsilon}{2}$ $\forall m \geq m_\varepsilon$

Hence $|x_m - x_n| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

$\forall n, m \geq n_\varepsilon$

i.e $\forall \varepsilon>0$ $\exists n_\varepsilon \in \mathbb{N} \ni$

$|x_n - x_m| < \varepsilon$ $\forall n, m \geq n_\varepsilon$

Thus, the sequence $\{x_n\}$ is Cauchy.

Q5b

State and prove the sandwich theorem

ANSWER

(sandwich theorem) suppose $\{a_n\}$ and $\{b_n\}$ are sequence of real numbers $\ni$ for some integer $N_0 \geq 1$

We have

$L \leq b_n \leq a_n \quad \forall n \geq N_0$

If $\{a_n\}$ converges to L then $\{b_n\}$ also converge to L

Proof by definition $a_n \to L$ as $n \to \infty$ i.e given $\varepsilon > 0$ $\exists$ $n_\varepsilon \in \mathbb{N} \ni |a_n - L| < \varepsilon \quad \forall n \geq n_\varepsilon$

Observe that $0 \leq b_n - L, \ \forall n \geq N_0$ so that since $b_n \leq a_n$ we have $0 \leq |b_n - L| \leq |a_n - L|$.

So given $\varepsilon > 0$ and all $n > \max\{n_\varepsilon, N_0\}$ we have $|b_n - L| \leq |a_n - L| < \varepsilon$

i.e $|b_n - L| < \varepsilon$

Hence, $b_n \to L$ as $n \to \infty$

i.e $0 < \frac{1}{2^n} \leq \frac{1}{n} \quad \frac{1}{n} \to 0$ also $\frac{1}{2^n} \to 0$

Hence, these complete the proof.

Q6a

The D'Alembert's ratio test state that for every $x \in \mathbb{R}$ and $x_n \neq 0 \ \forall n$ and $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = L \ \forall n$ satisfy IF:

(i) $L < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent

(ii) $L > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent

(iii) $L = 1$, then test Fail

(iv) $\forall x \in \mathbb{R}$ the series $\sum_{n=1}^{\infty} \frac{x^n}{x!}$ converges

proof: Here $x_n = \frac{x^n}{n!}$ and $x_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\frac{x_{n+1}}{x_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}$$

$$= \frac{x^n \cdot x}{n!(n+1)} \times \frac{n!}{x^n} = \frac{x}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \frac{x}{n+1} = 0 \Rightarrow L < 1$$

Hence the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is absolutely convergent.

Q6B

States Bolzano-weierstrass theorem without prove?

ANSWER

Bolzano-weierstrass theorem states that Every bounded sequence of real numbers is a convergent

Solution to 2018/2019 Test Question

Check the solution from the exam answer for all but 3b.

Does every bounded sequence converge?

b, Every convergent sequence is bounded but not every bounded sequence converge. Consider the sequence

$\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1 \ldots\}$, clearly sequence whose set of values is $\{-1, 1\}$ is bounded.

$\Rightarrow ((-1)^n)$ is bounded. Since $|x_n| = |(-1)^n| = 1 \ \forall n \geq 1$

$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N} \ \ni |(-1)^n - x| < \varepsilon \ \forall n \geq N(\varepsilon)$. Suppose that $(-1)^n$ converges to x, then

Suppose that $\varepsilon = \frac{1}{2}$ and $n = 1 \Rightarrow |(-1) - 1| = |-2| = 2$

$2 = |(-1)^n - (-1)^{n+1}| = |(-1)^n - x + x - (-1)^{n+1}| \leq |(-1)^n - x| + |(-1)^{n+1} - x|$

$< \frac{1}{2} + \frac{1}{2} = 1$ we say $2 = 1$ which is absurd

$\therefore$ The sequence $\{(-1)^n\}$ does not converge.

2019/2020 Test Solution.

Check the previous solution but Question 3(a).

Prove that $\sup\{\frac{n}{4n+1} : n = 1, 2, 3 \ldots\} = \frac{1}{4}$

Proof. Its sufficies to show that the 1st condition is sufficient.

1. $\frac{1}{4}$ is an upper bound of S i.e $\left(\frac{n}{4n+1} \leq \frac{1}{4}\right)$

2. $\forall \varepsilon > 0, \ \exists s \in S \ \ni \frac{1}{4} - \varepsilon < s \leq \frac{1}{4}$

Clearly, $\frac{n}{4n+1} < \frac{n}{4n} \leq \frac{1}{4} \ \forall n \geq 1$ So, $\frac{1}{4}$ is an upper bound of S

To verify (ii), let $\varepsilon > 0$ be abitrary, let us now find $s \in S \ \ni \frac{1}{4} - \varepsilon < s \leq \frac{1}{4}$ Since $s \in S$, it most have the form $\frac{n_0}{4n_0+1}$ for some positive integer n_0 i.e $\frac{1}{4} - \varepsilon < \frac{n_0}{4n_0+1} < \frac{1}{4}$

$\therefore \frac{n_0}{4n_0+1} < \frac{1}{4}$ holds $\forall$ integer n_0 but $\frac{1}{4} - \varepsilon < \frac{n_0}{4n_0+1}$ also hold $\forall \mathbb{Z}^+ n_0$ So $\frac{1-4\varepsilon}{4} < \frac{n_0}{4n_0+1}$ $(1-4\varepsilon)(4n_0+1) < 4n_0$

$4n_0 + 1 - 16\varepsilon n_0 - 4\varepsilon < 4n_0 \Rightarrow 16\varepsilon n_0 > 1 - 4\varepsilon \Rightarrow n_0 > \frac{1-4\varepsilon}{16\varepsilon}$

Hence our condition is satisfy.

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End of First Semester Examination.
Course: MATH 203, Real Analysis I,
Instruction: Attempt four questions only. Time: 2 hours.

1. (a) Prove that for any three real numbers x, y and d we have the following valid results;
   (i) $|x| < d$ implies that $-d < x < d$
   (ii) $|x + y| \leq |x| + |y|$.
   (iii) $|x| - |y| \leq |x - y|$
   (b) Find all $x \in \mathbb{R}$ that satisfy the inequality; $|x - 2| > |x + 1|$.

2. (a) Suppose that $S \subseteq \mathbb{R}$, $R \subseteq S$ and that $\forall x \in S$, there exist $r \in R$ such that $x \leq r$. Show that $\sup S = \sup R$.
   (b) Find the sup and inf of the following;
   (i) $\{\frac{n}{n+1}, n \in \mathbb{N}\}$, (ii) $\{\frac{(-1)^n n}{2n+1}, n \in \mathbb{N}\}$.

3. (a) Define a convergent sequence and show that the limit of a convergent sequence is unique
   (b) "Every convergent sequence is bounded". Is the converse of this statement true? If yes prove it, otherwise give a counter example.

4. (a) Define Completeness of a set X. Is the set $\mathbb{Q}$ of all rational numbers complete? Justify.
   (b) Define Cauchy sequence and prove that every convergent sequence is Cauchy sequence.

5. (a) Verify the convergence or divergence of the following series;
   (i) $\sum_{n=1}^{\infty} \frac{n}{n^2-3}$, (ii) $\sum_{n=1}^{\infty} \frac{n^2}{\exp(n^2)}$.
   (b) Is the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+4}$ convergent? Justify.

6. (a) Define Absolute and conditional convergence of a series and prove that an absolutely convergent series is a convergent series.
   (b) Determine which of the following series converges absolutely, converges conditionally or diverges:
   (i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (ii) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$

Solution To Exam Question

1. $|x| < d \iff -d < x < d$. Since $x = -|x|$ or $x = |x|$, it follows that $-|x| \le x \le |x|$ also $|x| < d$, then we have $-d \le -|x| \le x \le |x| \le d$. Conversely, Suppose $-d \le x \le d$. If $x \ge 0$, then $|x| = x < d$ and If $x < 0$ then $|x| = -x \le d$, combining both cases, we said $|x| \le d$

I. $|x+y| \le |x| + |y|$

Prove. The Inequality $-|x| \le x \le |x|$ are true since $x = -|x|$ or $x = |x|$ and $-|y| \le y \le |y|$ since $y = -|y|$ or $y = |y|$.
So we have $((-|x|)) + (-|y|) \le -|x| + y \le x + y \le |x| + |y|$
So $-(|x| + |y|) \le x+y \le |x| + |y| \Rightarrow x+y \le |x| + |y|$ and $-(x+y) \le |x| + |y|$ since $x+y = |x+y|$ or $-(x+y) = |x+y|$ we have $|x+y| \le |x| + |y|$.

III. $|x| - |y| \le |x-y|$

for Inequalities

$-|x| \le x \le |x|$ & $-|y| \le y \le |y|$ are true since $x = -|x|$ or $x = |x|$ and $y = -|y|$ or $y = |y|$ for $|x| = |x+y-y| \le |x-y| + |y| \Rightarrow |x| \le |x-y| + |y|$. $|x| - |y| \le |x-y|$ proved.

II. $|x-2| > |x+1|$ $\quad |x| > -c$ or $|x| > c$

From $|x| > c$ $\quad |x| > -c$

$x - 2 > x + 1$ or $x - 2 > -(x+1)$

$\cancel{x} > \cancel{x} + 3$ or $2x > 1$

$0 > 3$ — no solution

or $x > \frac{1}{2}$

Alternatively

Square both side

$(x-2)^2 > (x+1)^2$

i.e $(x-2)(x+2) > (x+1)(x+1)$

$\cancel{x^2} - 4x + 4 > \cancel{x^2} + 2x + 1$

$-6x > -3$

$x > \frac{3}{6}$

$x > \frac{1}{2}$

2. For $r \in R \ni x \le r$ and $S \subseteq R$, $R \subseteq S$ then $\sup S = \sup R$

It is sufficient to prove that the condition for supremum is satisfield.

let $\gamma = \sup S$

i) $x \le \gamma \ \forall x \in S$ (ii) $\forall \varepsilon > 0$ $\exists x \in S \ni \beta - \varepsilon \le x_0 \le \beta + \varepsilon$

These give rise to

i) $x \le \gamma \ \forall x \in R$ Since $S \subseteq R$

Then $\sup S \le \sup R$

Since they are proper subset we say $\sup S = \sup R$

These complete our prove.

b. $\{\frac{n}{n+1}, n \in N\}$. To find the supremum, find the limit the sequence converge to: $\lim_{n \to \infty} \frac{n}{n+1} = 1$

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Therefore the Sup is 1
Also, at $n=1$

$$x_n = \frac{n}{n+1} = \frac{1}{2}$$

Hence $\frac{1}{2}$ is the infimum (G(b))
i.e $[\frac{1}{2}, 1]$

ii) $x_n = \left\{ \frac{(-1)^n n}{2n+1}, n \in \mathbb{N} \right\} \forall n \geq 1$

$$x_n = \frac{(-1)^1 \cdot 1}{2+1} = \frac{-1}{3}$$

For the superemum, we find the limit as $n \to \infty$

$$\lim_{n \to \infty} \frac{(-1)^n \cdot n}{2n+1} = \infty$$

i.e It is not a convergent sequence but divergent. Hence no supremum

3. Check previous solution

4. A set is called complete if every Cauchy sequence is converge to an element in S
i.e $\frac{1}{n} \in (0, 1)$ is not complete.
but $\frac{1}{n} \in [0, 1]$ is complete.

b. the set Q of rational nos is not complete, we say let $\{a_n\} = (1 + \frac{1}{n})^n = e$ as $n \to \infty$
Since e is not in Q so, we say Q is not complete.
NOTE: Research more to see a complex solution.

5a. (i) $\sum_{n=1}^{\infty} \frac{n}{n^4 - 3}$

Test using D'Alembert ratio test

$$\lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right| = l \quad \forall n$$

Testing using Comparison theorem

$$x_n = \frac{n}{n^4 - 3}, \quad |x_n| = \left|\frac{n}{n^4 - 3}\right| < \left|\frac{n}{n^4}\right| = \left|\frac{1}{n^3}\right|$$

$$\sum_{n=1}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ — Divergent series}$$

Hence it gives an harmonic series

ii) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^2}}$

Using Cauchy root test

$$\lim_{n \to \infty} \sqrt[n]{|x_n|} = l \quad \forall n$$

$$|x_n| = \left|\frac{n^2}{e^{n^2}}\right| = \frac{n^2}{e^{n^2}}$$

$$\sqrt[n]{|x_n|} = \sqrt[n]{\frac{n^2}{e^{n^2}}} = \left(\frac{n^2}{e^{n^2}}\right)^{1/n} = \frac{n}{e^n}$$

$$\lim_{n \to \infty} \sqrt[n]{|x_n|} = \lim_{n \to \infty} \frac{n}{e^n} = 0 = l$$

Hence $l < 1$
then, the series is absolutely convergent.

6a. A series $\sum_{n=1}^{\infty} x_n$ of real nos is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |x_n|$ is convergent.

* A conditional convergence series states that if the series $\sum_{n=1}^{\infty}$ of real nos converge absolutely, then $\sum_{n=1}^{\infty} x_n$ converges.

6b.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

... in ... comparison test

$$|x_n| = \left|\frac{(-1)^n}{2^n}\right| < \left|\frac{1}{2^n}\right|$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \leq \sum_{n=1}^{\infty} |y_n|$$

Hence $x_n = \frac{1}{2^n}$ is an example of p-series which converge then $\sum_{n=1}^{\infty} |y_n|$ is absolutely convergent.

* Solution To Exercise given in the class

1) prove that the sequence (x_n) is null where x_n is given by

1. $\frac{3n+2}{n^2+1}$ (ii) $\frac{n^2+4}{n^3-12}$ (iii) $\frac{(-1)^n}{\sqrt{n}}$

(iv) $\frac{n^3+2n^2-1}{n^4-n^2+2}$

Solution

To prove these, we find the limit of the sequence as $x_n \to \infty$ if the limit tend to infinity, then the sequence is null.

1) $x_n = \frac{3n+2}{n^2+1}$

To do these, divide everything by the highest power

$$\lim_{n\to\infty} \frac{3n+2}{n^2+1} = \frac{\frac{3n}{n^2} + \frac{2}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}$$

$$= \frac{\frac{3}{\infty} + \frac{2}{\infty^2}}{1 + \frac{1}{\infty^2}} = \frac{0 + 0}{1 + 0} = \infty$$

Then the limit is null.

2) $x_n = \frac{n^2+4}{n^3-12}$. Do the same thing.

$$\lim_{n\to\infty} \frac{n^2+4}{n^3-12} \quad \frac{\frac{n^2}{n^3} + \frac{4}{n^3}}{\frac{n^3}{n^3} - \frac{12}{n^3}}$$

$$n \to \infty \quad \frac{\frac{1}{n} + \frac{4}{n^3}}{1 - \frac{12}{n^3}} = \frac{\frac{1}{\infty} + \frac{4}{\infty^2}}{1 - \frac{12}{\infty^3}} \quad \frac{0}{1}$$

Hence the $x_n = \frac{n^2+4}{n^3-12}$ is null.

*Try question iii and iv.

2) prove that the following sequence converge.

(i) $\left\{\frac{n-1}{n+1}\right\}$ (ii) $\left\{\frac{3n^3-3n^2-n-1}{n^3+n^2-2}\right\}$ $(n \geq 2)$

iii) $\left\{\sqrt{n+1} - \sqrt{n}\right\}$ (iv) $\left\{\frac{3n^2-1}{n^2-5n}\right\}$ $(n \geq 6)$

Solution

First test the limit to know where the sequence converge to

1) $x_n = \frac{n-1}{n+1}$ $\lim_{n\to\infty} \frac{n-1}{n+1}$ converges to 1 by dividing by the highest power

Now, prove that n is depending on epsilon.

From the definition of convergent series.

Let $\varepsilon > 0$ $\exists N(\varepsilon) \in \mathbb{N}$ $\ni |x_n - x| < \varepsilon$ $\forall n \geq N(\varepsilon)$

Then $\left|\frac{n-1}{n+1} - 1\right| < \varepsilon$

$$\left|\frac{n-1-(n+1)}{n+1}\right| = \left|\frac{-2}{n+1}\right| < \left|\frac{2}{n+1}\right| < \varepsilon$$

$$2 < (n+1)\varepsilon$$

$$2 - \varepsilon < n\varepsilon$$

$$n > \frac{2-\varepsilon}{\varepsilon}$$

Hence the sequence converge, since it is depending on epsilon.

2) $x_n = \sqrt{n+1} - \sqrt{n}$

Take the limit

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \sqrt{\frac{n}{n}+\frac{1}{n}} - \sqrt{\frac{n}{n}}$$

$$\sqrt{1+\frac{1}{\infty}} - \sqrt{1} = 0$$

Hence the limit converge to 0

Now to prove that $n(\varepsilon)$

$\exists N(\varepsilon) \in \mathbb{N} \ni |x_n - x| < \varepsilon$

let $\varepsilon > 0$

$|\sqrt{n+1} - \sqrt{n} - 0| < \varepsilon$

Rationalize the $x_n = \sqrt{n+1} - \sqrt{n}$

we have $\sqrt{n+1} - \sqrt{n} \times \dfrac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$

$$\left|\frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}}\right| < \left|\frac{1}{\sqrt{n}+\sqrt{n}}\right| = \left|\frac{1}{2\sqrt{n}}\right| < \varepsilon$$

$$\frac{1}{2\sqrt{n}} < \varepsilon \qquad 1 < 2\sqrt{n}\,\varepsilon$$

$$\sqrt{n} < \frac{1}{2\varepsilon}$$

$$n > \frac{1}{4\varepsilon^2}$$

Hence it converge. and use the same method and try the remain one.

3) Determine which of the following are monotonic and hence where applicable guess there limit

1. $\left\{\dfrac{3n+2}{2n-5}\right\}$ (ii) $\dfrac{n^2+1}{n}$

For quotient

$\dfrac{x_{n+1}}{x_n} \geq 1$

$x_n = \dfrac{3n+2}{2n-5}$, $x_{n+1} = \dfrac{3(n+1)+2}{2(n+1)-5}$

$$x_{n+1} = \frac{3n+5}{2n-3}$$

$$\frac{x_{n+1}}{x_n} = \frac{3n+5}{2n-3} \div \frac{3n+2}{2n-5}$$

$$\frac{3n+5}{2n-3} \times \frac{2n-5}{3n+2}$$

$$= \frac{6n^2-15n+10n-25}{6n^2+4n-6n-6}$$

$$= \frac{6n^2-5n-25}{6n^2-2n-6}$$

Using polynomial division

$$6n^2-2n-6 \,\overline{\big)\, 6n^2-5n-25}$$
quotient 1

$$\frac{6n^2-2n-6}{3n+19}$$

$$1 + \frac{3n+19}{6n^2-2n-6}$$

Since the $\dfrac{x_{n+1}}{x_n} \geq 1$

Hence it is monotone increasing sequence and the limit of the sequence is $\frac{3}{2}$

* Try No 2 question.

1) Test for Absolutely convergent of the following

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2}$

Testing using Alembert ratio

$x_n \neq 0\ \forall n$ and $\lim_{n\to\infty} \left|\dfrac{x_{n+1}}{x_n}\right| = L$

$$x_n = \frac{n!}{n^2} \qquad x_{n+1} = \frac{(n+1)!}{(n+1)^2}$$

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)!}{(n+1)^2} \div \frac{n!}{n^2}$$

$$\frac{(n+1)!}{(n+1)^2} \times \frac{n^2}{n!}$$

$$\frac{n!(n+1)}{(n+1)^2} \times \frac{n^2}{n!} = \frac{n^2}{n+1}$$

Hence, the limit ~~diverge~~ converge to infinity, i.e $l > 1$

If $l > 1$, then we say it is a divergent series.

ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

Using comparison test

$x_n = \dfrac{n^2}{n^3+1} < \left|\dfrac{n^2}{n^3}\right| \div \left|\dfrac{1}{n}\right| = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is also a divergent series.

* Try these yourself

i) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n^2}{n+1}$ (ii) $\displaystyle\sum \frac{1}{n+6}$

iii) $\displaystyle\sum \frac{3^n}{n+2}$

5) prove that $\left(\dfrac{1}{\sqrt{n}}\right)$ converges and find its limit

Solution

observe that $\dfrac{a_{n+1}}{a_n}$; $a_n = \dfrac{1}{\sqrt{n}}$

$a_{n+1} = \dfrac{1}{\sqrt{n+1}}$

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$$\frac{x_{n+1}}{x_n} = \frac{1}{\sqrt{n+1}} \div \frac{1}{\sqrt{n}}$$

$$\frac{x_{n+1}}{x_n} = \sqrt{\frac{n}{n+1}} < 1 \quad \forall n \geq 1$$

i.e $a_n \geq a_{n+1}$ $\forall n \geq 1$.

So $\frac{1}{\sqrt{n}}$ is a monotone decreasing se[quence] bounded below by 0 is there[fore] [se]quence) $\lim_{n\to\infty} \frac{1}{\sqrt{n}}$ exists

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, for $\left|\frac{1}{\sqrt{n}} - 0\right|$

$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} < \varepsilon$

$n > \frac{1}{\varepsilon^2}$, So given $\varepsilon > 0$ choose N, $n(\varepsilon) = \left[\frac{1}{\varepsilon^2}\right] + 1$

Additional Solution From Exercise given in the class

1. Let $b_1 = 1$ and $b_{n+1} = \sqrt{2+b_n}$ Show that $\{b_n\}$ is Convergent and find limit.

Solution

From $b_1 = 1$, we obtain
$b_2 = \sqrt{2+b_1} = \sqrt{3}$, $b_3 = \sqrt{2+b_2} = \sqrt{2+\sqrt{3}} \approx \sqrt{2+1.732}$,
and so on. Hence, we suspect that the sequence is monotone increasing and try to prove this by Induction.

Claim 1. $b_n \leq b_{n+1}$ $\forall n \geq 1$
Clearly the claim holds $\forall n = 1$, since $b_1 = 1 < \sqrt{3} = b_2$
Assume it also holds $\forall n = k$ i.e ($b_k \leq b_{k+1}$) then,

$$b_{k+2} = \sqrt{2+b_{k+1}} \geq \sqrt{2+b_k} = b_{k+1}$$

Hence, by induction, $\{b_n\}$ is monotone increasing

Claim 2. $b_n \leq 2$ For all $n \geq 1$
The proof of this claim is also by induction. For $n = 1$, $b_1 = 1 < 2$
Assume $b_k \leq 2$ For some integer $k > 0$, Then $b_{k+1} = \sqrt{2+b_k} \leq \sqrt{4} = 2$.
Hence, by induction $\{b_n\}$ is bounded above. i.e $\{b_n\}$ converges.
Let $b_n = x$. Then, since $\lim b_n = \lim b_{n+1}$, we have $x = \sqrt{2+x}$ which yields $x^2 - x - 2 = 0$. Hence $x = \frac{1}{2} + \frac{\sqrt{5}}{2}$ or $x = \frac{1}{2} - \frac{\sqrt{5}}{2}$ since $b_1 = 1$ and $\{b_n\}$ is monotone increasing, $\lim b_n = \frac{1}{2}(1+\sqrt{5})$

2. Show that the sequence $\langle x_n \rangle$ defined by
$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ Converge. $\forall n \in \mathbb{N}$

Every bounded monotone sequence converge.

Then $x_{n+1} = \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots \frac{1}{(n+1)+(n+1)}$

$x_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2(n+1)}$

$x_{n+1} > x_n$ $\forall n$

Clearly, $x_2 > x_1$, $x_3 > x_2$, $x_4 > x_3$

$x_1 < x_2 < x_3 \cdots$ i.e $x_n < x_{n+1} < x_{n+2} \cdots$

$\langle x_n \rangle$

$|x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+n} < \frac{1}{n} + \frac{1}{n} + \cdots \frac{1}{n} = \frac{n}{n} = 1$

$|x_n| < 1$ $\therefore \{x_n\}$ is bounded and also conver[gent]

~~$n < n+1 \Rightarrow \frac{1}{n+1} < \frac{1}{n}$~~

~~$n < n+2 \Rightarrow \frac{1}{n+2} < \frac{1}{n}$~~

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3. Test the Convergence or otherwise of the following series

I. $\sum_{n=1}^{\infty} \frac{10^n}{n}$ (II) $\sum_{n=1}^{\infty} \frac{\log^n}{2^n}$ (III) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ (IV) $\sum_{n=1}^{\infty} \frac{a^n}{n^2}, a>0$

Solution

I) If $x_n = \frac{10^n}{n}$ $x_{n+1} = \frac{10^{n+1}}{n+1}$, using ratio test $\forall x_n \neq 0$

$\left|\frac{x_{n+1}}{x_n}\right| = l$ $\therefore \frac{x_{n+1}}{x_n} = \frac{10^{n+1}}{n+1} \div \frac{10^n}{n}$

$$= \frac{10^{n+1}}{n+1} \times \frac{n}{10^n} \Rightarrow \frac{10^n 10}{n+1} \times \frac{n}{10^n}$$

$$= \frac{10n}{n+1} \to \frac{10}{1} \text{ as } n \to \infty$$

Hence $\sum_{n=1}^{\infty} x_n$ diverges by ratio test since $l > 1$

II) $\sum_{n=1}^{\infty} \frac{\log^n}{2^n}$ Test using cauchy root test

$\forall x_n \neq 0 \ \forall n \ \lim_{n\to\infty} \sqrt[n]{|x_n|} = l$

$$\left|\frac{\log n}{2^n}\right| = \frac{\log n}{2^n}$$

$$\sqrt[n]{\frac{\log n}{2^n}} = \frac{\sqrt[n]{\log n}}{\sqrt[n]{2^n}} = \frac{\sqrt[n]{\log n}}{2}$$

Clearly, $\sqrt[n]{\log n}$ converges to 1 as $n \to \infty$

Therefore $\sum_{n=1}^{\infty} \frac{\log n}{2^n}$ converges to $\frac{1}{2}$ as $n \to \infty$

Since $\frac{1}{2} < 1$ by Cauchy.

III) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ Applying the Cauchy's root test,

$$\sqrt[n]{|x_n|} = \sqrt[n]{\frac{n}{2^n}} = \frac{\sqrt[n]{n}}{\sqrt[n]{2^n}} = \frac{\sqrt[n]{n}}{2} \to \frac{1}{2} \text{ as } n \to \infty$$

Since $l < 1$

$\sum_{n=1}^{\infty} x_n = \frac{n}{2^n}$ is absolutely Convergent.

4) Verify if the sequence define by

I. $x_n = 1 - \frac{1}{n}$ (II) $x_n = \frac{1}{n^2}$ (III) $x_n = n^3$

are monotone increasing, monotone decreasing or not

Solution

I. $x_n = 1 - \frac{1}{n}$ & $x_{n+1} = 1 - \frac{1}{n+1}$

Examine the difference

$$x_{n+1} - x_n = 1 - \frac{1}{n+1} - 1 + \frac{1}{n}$$

$$\frac{-n+n+1}{n(n+1)} = \frac{1}{n(n+1)} \geq 0$$

Since $x_{n+1} - x_n \geq 0 \ \forall n \in \mathbb{N}$ then the sequence is monotone increasing.

II. $x_n = \frac{1}{n^2}$ & $x_{n+1} = \frac{1}{(n+1)^2}$

$$\frac{x_{n+1}}{x_n} = \frac{1}{(n+1)^2} \div \frac{1}{n^2}$$

$$= \frac{1}{(n+1)^2} \times \frac{n^2}{1} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2$$

Use polynomial division

$$\begin{array}{r|l} & 1 \\ n+1 & n \\ & -(n+1) \\ \hline & -1 \end{array}$$

$\therefore 1 - \frac{1}{n+1}$

$$\therefore \left(\frac{n}{n+1}\right)^2 = \left(1 - \frac{1}{n+1}\right)^2 = 1 - \frac{2}{n+1} + \frac{1}{(n+1)^2} \leq 1$$

Since $\frac{x_{n+1}}{x_n} \leq 1 \ \forall n \in \mathbb{N}$ then the sequence is monotone decreasing

III.) $x_n = n^3$, $x_{n+1} = (n+1)^3 = 3n^2 + 3n + n^3 + 1$

$$x_{n+1} - x_n = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1$$

$$3n^2 + 3n + 1 > 0 \ \forall n \in \mathbb{N}$$

Hence the sequence defined by $\{x_n\} = \{n^3\}$ is strictly monotone increasing.